Differential Forms in Electromagnetics

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Differential forms can be fun. Snapshot at the time of the 1978 URSI General Assembly in Helsinki Finland, showing Professor Georges A. Deschamps and the author disguised in fashionable sideburns.

This treatise is dedicated to the memory of Professor Georges A. Deschamps (1911–1998), the great proponent of differential forms to electromagnetics. He introduced this author to differential forms at the University of Illinois, Champaign-Urbana, where the latter was staying on a postdoctoral fellowship in 1972–1973. Actually, many of the dyadic operational rules presented here for the first time were born during that period. A later article by Deschamps [18] has guided this author in choosing the present notation.

Contents

Preface

The present text attempts to serve as an introduction to the differential form formalism applicable to electromagnetic field theory. A glance at Figure 1.2 on page 18, presenting the Maxwell equations and the medium equation in terms of differential forms, gives the impression that there cannot exist a simpler way to express these equations, and so differential forms should serve as a natural language for electromagnetism. However, looking at the literature shows that books and articles are almost exclusively written in Gibbsian vectors. Differential forms have been adopted to some extent by the physicists, an outstanding example of which is the classical book on gravitation by Misner, Thorne and Wheeler [58].

The reason why differential forms have not been used very much may be that, to be powerful, they require a toolbox of operational rules which so far does not appear to be well equipped. To understand the power of operational rules, one can try to imagine working with Gibbsian vectors without the bac cab rule $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$ which circumvents the need of expanding all vectors in terms of basis vectors. Differential-form formalism is based on an algebra of two vector spaces with a number of multivector spaces built upon each of them. This may be confusing at first until one realizes that different electromagnetic quantities are represented by different (dual) multivectors and the properties of the former follow from those of the latter. However, multivectors require operational rules to make their analysis effective. Also, there arises a problem of notation because there are not enough fonts for each multivector species. This has been solved here by introducing marking symbols (multihooks and multiloops), easy to use in handwriting like the overbar or arrow for marking Gibbsian vectors. It was not typographically possible to add these symbols to equations in the book. Instead, examples of their use have been given in figures showing some typical equations. The coordinate-free algebra of dyadics, which has been used in conjunction with Gibbsian vectors (actually, dyadics were introduced by J.W. Gibbs himself in the 1880s, [26–28]), has so far been missing from the differential-form formalism. In this book one of the main features is the introduction of an operational dyadic toolbox. The need is seen when considering problems involving general linear media which are defined by a set of medium dyadics. Also, some quantities which are represented by Gibbsian vectors become dyadics in differential-form representation. A collection of rules for multivectors and dyadics is given as an appendix at the end of the book. An advantage of differential forms when compared to Gibbsian vectors often brought forward lies in the geometrical content of different (dual) multivectors, best illustrated in the aforementioned book on gravitation. However, in the present book, the analytical aspect is emphasized because geometrical interpretations do not help very much in problem solving. Also, dyadics cannot be represented geometrically at all. For complex vectors associated with time-harmonic fields the geometry becomes complex.

It is assumed that the reader has a working knowledge on Gibbsian vectors and, perhaps, basic Gibbsian dyadics as given in [40]. Special attention has been made to introduce the differential-form formalism with a notation differing from that of Gibbsian notation as little as possible to make a step to differential forms manageable. This means balancing between notations used by mathematicians and electrical engineers in favor of the latter. Repetition of basics has not been avoided. In particular, dyadics will be introduced twice, in Chapters 1 and 2. The level of applications to electromagnetics has been left somewhat abstract because otherwise it would need a book of double or triple this size to cover all the aspects usually presented in books with Gibbsian vectors and dyadics. It is hoped such a book will be written by someone. Many details have been left as problems, with hints and solutions to some of them given as an appendix.

The text is an outgrowth of lecture material presented in two postgraduate courses at the Helsinki University of Technology. This author is indebted to two collaborators of the courses, Dr. Pertti Lounesto (a world-renown expert in Clifford algebras who sadly died during the preparation of this book) from Helsinki Institute of Technology, and Professor Bernard Jancewicz, from University of Wroclaw. Also thanks are due to the active students of the courses, especially Henrik Wallén. An early version of the present text has been read by professors Frank Olyslager (University of Ghent) and Kurt Suchy (University of Düsseldorf) and their comments have helped this author move forward.

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1 Multivectors

1.1 THE GRASSMANN ALGEBRA

The exterior algebra associated with differential forms is also known as the Grassmann algebra. Its originator was Hermann Grassmann (1809–1877), a German mathematician and philologist who mainly acted as a high-school teacher in Stettin (presently Szczecin in Poland) without ever obtaining a university position.[1] His father, Justus Grassmann, also a high-school teacher, authored two textbooks on elementary mathematics, *Raumlehre* (*Theory of the Space*, 1824) and *Trigonometrie* (1835). They contained footnotes where Justus Grassmann anticipated an algebra associated with geometry. In his view, a parallelogram was a geometric product of its sides whereas a parallelepiped was a product of its height and base parallelogram. This must have had an effect on Hermann Grassmann's way of thinking and eventually developed into the algebra carrying his name.

In the beginning of the 19th century, the classical analysis based on Cartesian coordinates appeared cumbersome for many simple geometric problems. Because problems in planar geometry could also be solved in a simple and elegant way in terms of complex variables, this inspired a search for a three-dimensional complex analysis. The generalization seemed, however, to be impossible.

To show his competence for a high-school position, Grassmann wrote an extensive treatise(over 200 pages), *Theorie der Ebbe und Flut* (*Theory of Tidal Movement*, 1840). There he introduced a geometrical analysis involving addition and differentiation of oriented line segments (Strecken), or vectors in modern language. By

[1]This historical review is based mainly on reference 15. See also references 22, 37 and 39.

generalizing the idea given by his father, he defined the geometrical product of two vectors as the area of a parallelogram and that of three vectors as the volume of a parallelepiped. In addition to the geometrical product, Grassmann defined also a linear product of vectors (the dot product). This was well before the famous day, Monday October 16, 1843, when William Rowan Hamilton (1805-1865) discovered the four-dimensional complex numbers, the quaternions.

During 1842–43 Grassmann wrote the book *Lineale Ausdehnungslehre* (*Linear Extension Theory*, 1844), in which he generalized the previous concepts. The book was a great disappointment: it hardly sold at all, and finally in 1864 the publisher destroyed the remaining stock of 600 copies. *Ausdehnungslehre* contained philosophical arguments and thus was extremely hard to read. This was seen from the fact that no one would write a review of the book. Grassmann considered algebraic quantities which could be numbers, line segments, oriented areas, and so on, and defined 16 relations between them. He generalized everything to a space of n dimensions, which created more difficulties for the reader.

The geometrical product of the previous treatise was renamed as outer product. For example, in the outer product ab of two vectors (line segments) a and b the vector a was moved parallel to itself to a distance defined by the vector b, whence the product ab defined a parallelogram with an orientation. The orientation was reversed when the order was reversed: $ab = -ba$. If the parallelogram ab was moved by the vector c, the product abc gave a parallelepiped with an orientation. The outer product was more general than the geometric product, because it could be extended to a space of n dimensions. Thus it could be applied to solving a set of linear equations without a geometric interpretation.

During two decades the scientific world took the *Ausdehnungslehre* with total silence, although Grassmann had sent copies of his book to many well-known mathematicians asking for their comments. Finally, in 1845, he had to write a summary of his book by himself.

Only very few scientists showed any interest during the 1840s and 1850s. One of them was Adhemar-Jean-Claude de Saint-Venant, who himself had developed a corresponding algebra. In his article "Sommes et différences géométriques pour simplifier la mecanique" (Geometrical sums and differences for the simplification of mechanics, 1845), he very briefly introduced addition, subtraction, and differentiation of vectors and a similar outer product. Also, Augustin Cauchy had in 1853 developed a method to solve linear algebraic equations in terms of anticommutative elements ($ij = -ji$), which he called "clefs algébraiques" (algebraic keys). In 1852 Hamilton obtained a copy of Grassmann's book and expressed first his admiration which later turned to irony ("the greater the extension, the smaller the intention"). The afterworld has, however, considered the *Ausdehnungslehre* as a first classic of linear algebra, followed by Hamilton's book *Lectures on Quaternions* (1853).

During 1844–1862 Grassmann authored books and scientific articles on physics, philology (he is still a well-known authority in Sanscrit) and folklore (he published a collection of folk songs). However, his attempts to get a university position were not succesful, although in 1852 he was granted the title of Professor. Eventually, Grassmann published a completely rewritten version of his book, *Vollständige Aus-*

Maxwell		Gibbsian
$a = \frac{dH}{dy} - \frac{dG}{dz}$ $b = \frac{dF}{dz} - \frac{dH}{dx}$ $c = \frac{dG}{dx} - \frac{dF}{dy}$	(A)	$\mathbf{B} = \nabla \times \mathbf{A}$
$P = c\frac{dy}{dt} - b\frac{dz}{dt} - \frac{dF}{dt} - \frac{d\psi}{dx}$ $Q = a\frac{dz}{dt} - c\frac{dx}{dt} - \frac{dG}{dt} - \frac{d\psi}{dy}$ $R = b\frac{dx}{dt} - a\frac{dy}{dt} - \frac{dH}{dt} - \frac{d\psi}{dz}$	(B)	$\mathbf{E} = \mathbf{v} \times \mathbf{B} - \frac{\partial \mathbf{A}}{\partial t} - \nabla\phi$
$X = vc - wb$ $Y = wa - uc$ $Z = ub - va$	(C)	$\mathbf{F} = \mathbf{J} \times \mathbf{B}$
$a = \alpha + 4\pi A$ $b = \beta + 4\pi B$ $c = \gamma + 4\pi C$	(D)	$\mathbf{B} = \mu_o \mathbf{H} + \mathbf{M}$
$4\pi u = \frac{d\gamma}{dy} - \frac{d\beta}{dz}$ $4\pi v = \frac{d\alpha}{dz} - \frac{d\gamma}{dx}$ $4\pi w = \frac{d\beta}{dx} - \frac{d\alpha}{dy}$	(E)	$\mathbf{J} = \nabla \times \mathbf{H}$
$\mathfrak{D} = \frac{1}{4\pi} K \mathfrak{E}$	(F)	$\mathbf{D} = \epsilon \mathbf{E}$
$\mathfrak{K} = C\mathfrak{E}$	(G)	$\mathbf{J}_c = \sigma \mathbf{E}$
$\mathfrak{C} = \mathfrak{K} + \dot{\mathfrak{D}}$	(H)	$\mathbf{J} = \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t}$
$u = p + \frac{df}{dt}$ $v = q + \frac{dg}{dt}$ $w = r + \frac{dh}{dt}$	(H*)	$\mathbf{J} = \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t}$
$\mathfrak{C} = (C + \frac{1}{4\pi} K \frac{d}{dt})\mathfrak{E}$	(I)	$\mathbf{J} = \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t}$
$u = CP + \frac{1}{4\pi} K \frac{dP}{dt}$ $v = CQ + \frac{1}{4\pi} K \frac{dQ}{dt}$ $w = CR + \frac{1}{4\pi} K \frac{dR}{dt}$	(I*)	$\mathbf{J} = \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t}$
$\rho = \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}$	(J)	$\varrho = \nabla \cdot \mathbf{D}$
$\sigma = lf + mg + nh + l'f' + m'g' + n'h'$	(K)	$\varrho_s = \mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2)$
$\mathfrak{B} = \mu \mathfrak{H}$	(L)	$\mathbf{B} = \mu \mathbf{H}$

Fig. 1.1 *The original set of equations (A)–(L) as labeled by Maxwell in his Treatise (1873), with their interpretation in modern Gibbsian vector notation. The simplest equations were also written in vector form.*

dehnungslehre (*Complete Extension Theory*), on which he had started to work in 1854. The foreword bears the date 29 August 1861. Grassmann had it printed on his own expense in 300 copies by the printer Enslin in Berlin in 1862 [29]. In its preface he complained the poor reception of the first version and promised to give his arguments in Euclidean rigor in the present version.[2] Indeed, instead of relying on philosophical and physical arguments, the book was based on mathematical theorems. However, the reception of the second version was similar to that of the first one. Only in 1867 Hermann Hankel wrote a comparative article on the Grassmann algebra and quaternions, which started an interest in Grassmann's work. Finally there was also growing interest in the first edition of the *Ausdehnungslehre*, which made the publisher release a new printing in 1879, after Grassmann's death. Toward the end of his life, Grassmann had, however, turned his interest from mathematics to philology, which brought him an honorary doctorate among other signs of appreciation.

Although Grassmann's algebra could have become an important new mathematical branch during his lifetime, it did not. One of the reasons for this was the difficulty in reading his books. The first one was not a normal mathematical monograph with definitions and proofs. Grassmann gave his views on the new concepts in a very abstract way. It is true that extended quantities (Ausdehnungsgrösse) like multivectors in a space of n dimensions were very abstract concepts, and they were not easily digestible. Another reason for the poor reception for the Grassmann algebra is that Grassmann worked in a high school instead of a university where he could have had a group of scientists around him. As a third reason, we might recall that there was no great need for a vector algebra before the the arrival of Maxwell's electromagnetic theory in the 1870s, which involved interactions of many vector quantites. Their representation in terms of scalar quantites, as was done by Maxwell himself, created a messy set of equations which were understood only by a few scientist of his time (Figure 1.1).

After a short success period of Hamilton's quaternions in 1860-1890, the vector notation created by J. Willard Gibbs (1839–1903) and Oliver Heaviside (1850–1925) for the three-dimensional space overtook the analysis in physics and electromagnetics during the 1890s. Einstein's theory of relativity and Minkowski's space of four dimensions brought along the tensor calculus in the early 1900s. William Kingdon Clifford (1845–1879) was one of the first mathematicians to know both Hamilton's quaternions and Grassmann's analysis. A combination of these presently known as the Clifford algebra has been applied in physics to some extent since the 1930's [33, 54]. Élie Cartan (1869–1951) finally developed the theory of differential forms based on the outer product of the Grassmann algebra in the early 1900s. It was adopted by others in the 1930s. Even if differential forms are generally applied in physics, in electromagnetics the Gibbsian vector algebra is still the most common method of notation. However, representation of the Maxwell equations in terms of differential forms has remarkable simple form in four-dimensional space-time (Figure 1.2).

[2] This book was only very recently translated into English [29] based on an edited version which appeared in the collected works of Grassmann.

$$\begin{aligned} \mathbf{d} \wedge \mathbf{\Psi} &= \gamma \\ \mathbf{d} \wedge \mathbf{\Phi} &= 0 \\ \mathbf{\Psi} &= \overline{\overline{\mathsf{M}}}|\mathbf{\Phi} \end{aligned}$$

Fig. 1.2 *The two Maxwell equations and the medium equation in differential-form formalism. Symbols will be explained in Chapter 4.*

Grassmann had hoped that the second edition of *Ausdehnungslehre* would raise interest in his contemporaries. Fearing that this, too, would be of no avail, his final sentences in the foreword were addressed to future generations [15, 75]:

> ... But I know and feel obliged to state (though I run the risk of seeming arrogant) that even if this work should again remain unused for another seventeen years or even longer, without entering into actual development of science, still that time will come when it will be brought forth from the dust of oblivion, and when ideas now dormant will bring forth fruit. I know that if I also fail to gather around me in a position (which I have up to now desired in vain) a circle of scholars, whom I could fructify with these ideas, and whom I could stimulate to develop and enrich further these ideas, nevertheless there will come a time when these ideas, perhaps in a new form, will rise anew and will enter into living communication with contemporary developments. For truth is eternal and divine, and no phase in the development of the truth divine, and no phase in the development of truth, however small may be region encompassed, can pass on without leaving a trace; truth remains, even though the garments in which poor mortals clothe it may fall to dust.
>
> Stettin, 29 August 1861

1.2 VECTORS AND DUAL VECTORS

1.2.1 Basic definitions

Vectors are elements of an n-dimensional vector space denoted by $\mathbb{E}_1(n)$, and they are in general denoted by boldface lowercase Latin letters $\mathbf{a}$, $\mathbf{b}$,... Most of the analysis is applicable to any dimension n but special attention is given to three-dimensional Euclidean (Eu3) and four-dimensional Minkowskian (Mi4) spaces (these concepts will be explained in terms of metric dyadics in Section 2.5). A set of linearly independent vectors $\{\mathbf{e}_i\} = \mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$ forms a basis if any vector $\mathbf{a}$ can be uniquely expressed in terms of the basis vectors as

$$\mathbf{a} = \sum_{i=1}^{n} a_i \mathbf{e}_i, \tag{1.1}$$

where the a_i are scalar coefficients (real or complex numbers).

Dual vectors are elements of another n-dimensional vector space denoted by $\mathbb{F}_1(n)$, and they are in general denoted by boldface Greek letters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$,... A dual vector basis is denoted by $\{\boldsymbol{\varepsilon}_i\} = \boldsymbol{\varepsilon}_1,...,\boldsymbol{\varepsilon}_n$. Any dual vector $\boldsymbol{\alpha}$ can be uniquely expressed in terms of the dual basis vectors as

$$\boldsymbol{\alpha} = \sum_{i=1}^{n} \alpha_i \boldsymbol{\varepsilon}_i, \tag{1.2}$$

with scalar coefficients α_i. Many properties valid for vectors are equally valid for dual vectors and conversely. To save space, in obvious cases, this fact is not explicitly stated.

Working with two different types of vectors is one factor that distinguishes the present analysis from the classical Gibbsian vector analysis [28]. Vector-like quantities in physics can be identified by their nature to be either vectors or dual vectors, or, rather, multivectors or dual multivectors to be discussed below. The disadvantage of this division is, of course, that there are more quantities to memorize. The advantage is, however, that some operation rules become more compact and valid for all space dimensions. Also, being a multivector or a dual multivector is a property similar to the dimension of a physical quantity which can be used in checking equations with complicated expressions. One could include additional properties to multivectors, not discussed here, which make one step forward in this direction. In fact, multivectors could be distinguished as being either true or pseudo multivectors, and dual multivectors could be distinguished as true or pseudo dual multivectors [36]. This would double the number of species in the zoo of multivectors.

Vectors and dual vectors can be given geometrical interpretations in terms of arrows and sets of parallel planes, and this can be extended to multivectors and dual multivectors. Actually, this has given the possibility to geometrize all of physics [58]. However, our goal here is not visualization but developing analytic tools applicable to electromagnetic problems. This is why the geometric content is passed by very quickly.

1.2.2 Duality product

The vector space[3] $\mathbb{E}_1$ and the dual vector space $\mathbb{F}_1$ can be associated so that every element of the dual vector space $\mathbb{F}_1$ defines a linear mapping of the elements of the vector space $\mathbb{E}_1$ to real or complex numbers. Similarly, every element of the vector space $\mathbb{E}_1$ defines a linear mapping of the elements of the dual vector space $\mathbb{F}_1$. This mutual linear mapping can be expressed in terms of a symmetric product called the duality product (inner product or contraction) which, following Deschamps [18], is denoted by the sign $|$

$$\boldsymbol{\alpha},\ \mathbf{a} \rightarrow \boldsymbol{\alpha}|\mathbf{a} = \mathbf{a}|\boldsymbol{\alpha}. \tag{1.3}$$

A vector $\mathbf{a}$ and a dual vector $\boldsymbol{\alpha}$ can be called orthogonal (or, rather, annihilating) if they satisfy $\mathbf{a}|\boldsymbol{\alpha} = 0$. The vector and dual vector bases $\{\mathbf{e}_i\}$, $\{\boldsymbol{\varepsilon}_i\}$ are called

[3]When the dimension n is general or has an agreed value, iwe write $\mathbb{E}_1$ instead of $\mathbb{E}_1(n)$ for simplicity.

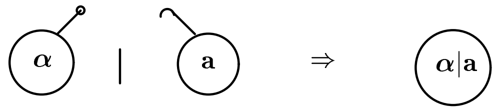

Fig. 1.3 *Hook and eye serve as visual aid to distinguish between vectors and dual vectors. The hook and the eye cancel each other in the duality product.*

reciprocal [21, 28] (dual in [18]) to one another if they satisfy

$$\boldsymbol{\varepsilon}_i|\mathbf{e}_j = \mathbf{e}_j|\boldsymbol{\varepsilon}_i = \delta_{ij}. \tag{1.4}$$

Here δ_{ij} is the Kronecker symbol, $\delta_{ij} = 0$ when $i \neq j$ and $= 1$ when $i = j$. Given a basis of vectors or dual vectors the reciprocal basis can be constructed as will be seen in Section 2.4. In terms of the expansions (1.1), (1.2) in the reciprocal bases, the duality product of a vector $\mathbf{a}$ and a dual vector $\boldsymbol{\alpha}$ can be expressed as

$$\boldsymbol{\alpha}|\mathbf{a} = \sum_{i,j}(\alpha_i\boldsymbol{\varepsilon}_i)|(a_j\mathbf{e}_j) = \sum_i \alpha_i a_i. \tag{1.5}$$

The duality product must not be mistaken for the scalar product (dot product) of the vector space, denoted by $\mathbf{a}\cdot\mathbf{b}$, to be introduced in Section 2.5. The elements of the duality product are from two different spaces while those of the dot product are from the same space.

To distinguish between different quantities it is helpful to have certain suggestive mental aids, for example, hooks for vectors and eyes for dual vectors as in Figure 1.3. In the duality product the hook of a vector is fastened to the eye of the dual vector and the result is a scalar with neither a hook nor an eye left free. This has an obvious analogy in atoms forming molecules.

1.2.3 Dyadics

Linear mappings from a vector to a vector can be conveniently expressed in the coordinate-free dyadic notation. Here we consider only the basic notation and leave more detailed properties to Chapter 2. Dyadic product of a vector $\mathbf{c}$ and a dual vector $\boldsymbol{\gamma}$ is denoted by $\mathbf{c}\boldsymbol{\gamma}$. The "no-sign" dyadic multiplication originally introduced by Gibbs [28, 40] is adopted here instead of the sign $\otimes$ preferred by the mathematicians. Also, other signs for the dyadic product have been in use since Gibbs,— for example, the colon [53].

The dyadic product can be defined by considering the expression

$$\mathbf{b} = \mathbf{c}(\boldsymbol{\gamma}|\mathbf{a}) = (\mathbf{c}\boldsymbol{\gamma})|\mathbf{a}, \tag{1.6}$$

which extends the associative law (order of the two multiplications as shown by the brackets). The dyad $\mathbf{c}\boldsymbol{\gamma}$ acts as a linear mapping from a vector $\mathbf{a}$ to another vector $\mathbf{b}$.

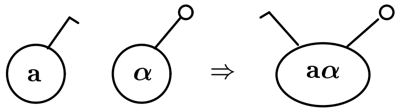

Fig. 1.4 *Dyadic product (no sign) of a vector and a dual vector in this order produces an object which can be visualized as having a hook on the left and an eye on the right.*

Similarly, the dyadic product $\boldsymbol{\gamma}\mathbf{c}$ acts as a linear mapping from a dual vector $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$ as

$$\boldsymbol{\beta} = \boldsymbol{\gamma}(\mathbf{c}|\boldsymbol{\alpha}) = (\boldsymbol{\gamma}\mathbf{c})|\boldsymbol{\alpha}. \tag{1.7}$$

The dyadic product $\mathbf{a}\boldsymbol{\alpha}$ can be pictured as an ordered pair of quantities glued back-to-back so that the hook of the vector $\mathbf{a}$ points to the left and the eye of the dual vector $\boldsymbol{\alpha}$ points to the right (Figure 1.4).

Any linear mapping within each vector space $\mathbb{E}_1$ and $\mathbb{F}_1$ can be performed through dyadic polynomials, or dyadics in short. Whenever possible, dyadics are denoted by capital sans-serif characters with two overbars, otherwise by standard symbols with two overbars:

$$\mathbf{b} = \overline{\overline{\mathsf{A}}}|\mathbf{a}, \quad \overline{\overline{\mathsf{A}}} = \sum \mathbf{c}_i\boldsymbol{\gamma}_i = \mathbf{c}_1\boldsymbol{\gamma}_1 + \cdots + \mathbf{c}_n\boldsymbol{\gamma}_n, \tag{1.8}$$

$$\boldsymbol{\beta} = \overline{\overline{\mathsf{A}}}^T|\boldsymbol{\alpha}, \quad \overline{\overline{\mathsf{A}}}^T = \sum \boldsymbol{\gamma}_i\mathbf{c}_i = \boldsymbol{\gamma}_1\mathbf{c}_1 + \cdots + \boldsymbol{\gamma}_n\mathbf{c}_n. \tag{1.9}$$

Here, ^{T} denotes the transpose operation: $(\mathbf{c}\boldsymbol{\gamma})^T = \boldsymbol{\gamma}\mathbf{c}$. Mapping of a vector by a dyadic can be pictured as shown in Figure 1.5.

Let us denote the space of dyadics of the type $\overline{\overline{\mathsf{A}}}$ above by $\mathbb{E}_1\mathbb{F}_1$ (short for $\mathbb{E}_1 \times \mathbb{F}_1$) and, that of the type $\overline{\overline{\mathsf{A}}}^T$ by $\mathbb{F}_1\mathbb{E}_1$ ($\mathbb{F}_1 \times \mathbb{E}_1$). An element of the space $\mathbb{E}_1\mathbb{F}_1$ maps the vector space $\mathbb{E}_1$ onto itself (from the right, from the left it maps the space $\mathbb{F}_1$ onto itself). If a given dyadic $\overline{\overline{\mathsf{A}}}$ maps the space $\mathbb{E}_1$ onto itself, i.e., any vector basis $\{\mathbf{e}_i\}$ to another vector basis $\{\mathbf{e}'_i\}$, the dyadic is called *complete* and there exists a unique inverse dyadic $\overline{\overline{\mathsf{A}}}^{-1}$. The dyadic is incomplete if it maps $\mathbb{E}_1$ only to a subspace of $\mathbb{E}_1$. Such a dyadic does not have a unique inverse. The dimensions of the dyadic spaces $\mathbb{F}_1\mathbb{E}_1$ and $\mathbb{E}_1\mathbb{F}_1$ are n^2.

The dyadic product does not commute. Actually, as was seen above, the transpose operation ^{T} maps dyadics $\mathbb{E}_1\mathbb{F}_1$ to another space $\mathbb{F}_1\mathbb{E}_1$. There are no concepts like symmetry and antisymmetry applicable to dyadics in these spaces. Later we will encounter other dyadic spaces $\mathbb{E}_1\mathbb{E}_1$, $\mathbb{F}_1\mathbb{F}_1$ containing symmetric and antisymmetric dyadics.

The unit dyadic $\overline{\overline{\mathsf{I}}}$ maps any vector to itself: $\overline{\overline{\mathsf{I}}}|\mathbf{a} = \mathbf{a}$. Thus, it also maps any $\mathbb{E}_1\mathbb{F}_1$ dyadic to itself: $\overline{\overline{\mathsf{I}}}|\overline{\overline{\mathsf{A}}} = \overline{\overline{\mathsf{A}}}$. Because any vector $\mathbf{a}$ can be expressed in terms of a basis $\{\mathbf{e}_i\}$ and its reciprocal dual basis $\{\boldsymbol{\varepsilon}_i\}$ as

$$\mathbf{a} = \sum \mathbf{e}_i(\boldsymbol{\varepsilon}_i|\mathbf{a}) = \sum(\mathbf{e}_i\boldsymbol{\varepsilon}_i)|\mathbf{a}, \tag{1.10}$$

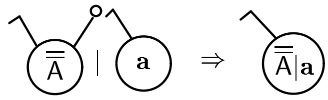

Fig. 1.5 *Dyadic* $\overline{\overline{\mathsf{A}}}$ *maps a vector* $\mathbf{a}$ *to the vector* $\overline{\overline{\mathsf{A}}}|\mathbf{a}$.

the unit dyadic can be expanded as

$$\overline{\overline{\mathsf{I}}} = \sum \mathbf{e}_i \varepsilon_i = \mathbf{e}_1\varepsilon_1 + \mathbf{e}_2\varepsilon_2 + \cdots + \mathbf{e}_n\varepsilon_n. \tag{1.11}$$

The form is not unique because we can choose one of the reciprocal bases $\{\mathbf{e}_i\}$, $\{\varepsilon_j\}$ arbitrarily. The transposed unit dyadic

$$\overline{\overline{\mathsf{I}}}^T = \sum \varepsilon_i \mathbf{e}_i = \varepsilon_1\mathbf{e}_1 + \varepsilon_2\mathbf{e}_2 + \cdots + \varepsilon_n\mathbf{e}_n \tag{1.12}$$

serves as the unit dyadic for the dual vectors satisfying $\overline{\overline{\mathsf{I}}}^T|\boldsymbol{\alpha} = \boldsymbol{\alpha}$ for any dual vector $\boldsymbol{\alpha}$. We can also write $\boldsymbol{\alpha}|\overline{\overline{\mathsf{I}}} = \boldsymbol{\alpha}$ and $\mathbf{a}|\overline{\overline{\mathsf{I}}}^T = \mathbf{a}$.

Problems

1.2.1 Given a basis of vectors $\{\mathbf{a}_i\}$ and a basis of dual vectors $\{\boldsymbol{\beta}_j\}$, find the basis of dual vectors $\{\boldsymbol{\alpha}_j\}$ dual to $\{\mathbf{a}_i\}$ in terms of the basis $\{\boldsymbol{\beta}_j\}$.

1.2.2 Show that, in a space of n dimensions, any dyadic $\overline{\overline{\mathsf{A}}}$ can be expressed as a sum of n dyads $\mathbf{a}_i\boldsymbol{\alpha}_i$.

1.3 BIVECTORS

1.3.1 Wedge product

The wedge product (outer product) between any two elements $\mathbf{a}$ and $\mathbf{b}$ of the vector space $\mathbb{E}_1$ and elements $\boldsymbol{\alpha}, \boldsymbol{\beta}$ of the dual vector space $\mathbb{F}_1$ is defined to satisfy the anticommutative law:

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}, \quad \boldsymbol{\alpha} \wedge \boldsymbol{\beta} = -\boldsymbol{\beta} \wedge \boldsymbol{\alpha}. \tag{1.13}$$

Anticommutativity implies that the wedge product of any element with itself vanishes:

$$\mathbf{a} \wedge \mathbf{a} = 0, \quad \boldsymbol{\alpha} \wedge \boldsymbol{\alpha} = 0. \tag{1.14}$$

Actually, (1.14) implies (1.13), because we can expand

$$(\mathbf{a} + \mathbf{b}) \wedge (\mathbf{a} + \mathbf{b}) = \mathbf{a} \wedge \mathbf{a} + \mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{a} + \mathbf{b} \wedge \mathbf{b}$$

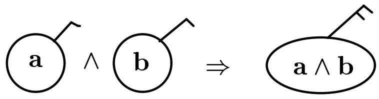

Fig. 1.6 *Visual aid for forming the wedge product of two vectors. The bivector has a double hook and, the dual bivector, a double eye.*

$$= \mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{a} = 0. \tag{1.15}$$

A scalar factor can be moved outside the wedge product:

$$\mathbf{a} \wedge (\lambda \mathbf{b}) = \lambda (\mathbf{a} \wedge \mathbf{b}). \tag{1.16}$$

Wedge product between a vector and a dual vector is not defined.

1.3.2 Basis bivectors

The wedge product of two vectors is neither a vector nor a dyadic but a bivector,[4] or 2-vector, which is an element of another space $\mathbb{E}_2$. Correspondingly, the wedge product of two dual vectors is a dual bivector, an element of the space $\mathbb{F}_2$. A bivector can be visualized by a double hook as in Figure 1.6 and, a dual bivector, by a double eye. Whenever possible, bivectors are denoted by boldface Roman capital letters like $\mathbf{A}$, and dual bivectors are denoted by boldface Greek capital letters like $\mathbf{\Phi}$. However, in many cases we have to follow the classical notation of the electromagnetic literature.

A bivector of the form $\mathbf{a} \wedge \mathbf{b}$ is called a simple bivector [33]. General elements of the bivector space $\mathbb{E}_2$ are linear combinations of simple bivectors,

$$\mathbf{A} = \mathbf{a}_1 \wedge \mathbf{b}_1 + \mathbf{a}_2 \wedge \mathbf{b}_2 + \cdots = \sum \mathbf{a}_i \wedge \mathbf{b}_i. \tag{1.17}$$

The basis elements in the spaces $\mathbb{E}_2$ and $\mathbb{F}_2$ can be expanded in terms of the respective basis elements of $\mathbb{E}_1$ and $\mathbb{F}_1$. The basis bivectors and dual bivectors are denoted by lowercase letters with double indices as

$$\mathbf{e}_{ij} = \mathbf{e}_i \wedge \mathbf{e}_j = -\mathbf{e}_{ji}, \tag{1.18}$$

$$\varepsilon_{ij} = \varepsilon_i \wedge \varepsilon_j = -\varepsilon_{ji}. \tag{1.19}$$

Due to antisymmetry of the wedge product, the bi-index ij has some redundancy since the basis elements with indices of the form ii are zero and the elements corresponding to the bi-index ij equal the negative of those with the bi-index ji. Thus, instead of n^2, the dimension of the spaces $\mathbb{E}_2(n)$ and $\mathbb{F}_2(n)$ is only $n(n-1)/2$. For the two-, three-, and four-dimensional vector spaces, the respective dimensions of the bivector spaces are one, three, and six.

[4]Note that, originally, J.W. Gibbs called complex vectors of the form $\mathbf{a} + j\mathbf{b}$ bivectors. This meaning is still occasionally encountered in the literature [9].

The wedge product of two vector expansions

$$\mathbf{a} = \sum a_i \mathbf{e}_i, \quad \mathbf{b} = \sum b_j \mathbf{e}_j \tag{1.20}$$

gives the bivector expansion

$$\mathbf{a} \wedge \mathbf{b} = \sum a_i \mathbf{e}_i \wedge \sum b_j \mathbf{e}_j = \sum_{i,j} a_i b_j \mathbf{e}_{ij} \tag{1.21}$$

$$= a_1 b_2 \mathbf{e}_{12} + a_2 b_1 \mathbf{e}_{21} + a_1 b_3 \mathbf{e}_{13} + a_3 b_1 \mathbf{e}_{31} + \cdots . \tag{1.22}$$

Because of the redundancy, we can reduce the number of bi-indices ij by ordering, i.e., restricting to indices satisfying $i < j$:

$$\begin{aligned} \mathbf{a} \wedge \mathbf{b} &= \sum_{i<j} (a_i b_j - a_j b_i) \mathbf{e}_{ij} \\ &= (a_1 b_2 - a_2 b_1) \mathbf{e}_{12} + (a_1 b_3 - a_3 b_1) \mathbf{e}_{13} + \cdots + (a_1 b_n - a_n b_1) \mathbf{e}_{1n} \\ &\quad + (a_2 b_3 - a_3 b_2) \mathbf{e}_{23} + (a_2 b_4 - a_4 b_2) \mathbf{e}_{24} + \cdots + (a_2 b_n - a_n b_2) \mathbf{e}_{2n} \\ &\quad + \cdots + (a_{n-1} b_n - a_n b_{n-1}) \mathbf{e}_{(n-1)n} . \end{aligned} \tag{1.23}$$

Euclidean and Minkowskian bivectors For a more symmetric representation, cyclic ordering of the bi-indices is often preferred in the three-dimensional Euclidean Eu3 space:

$$\mathbf{a} \wedge \mathbf{b} = (a_1 b_2 - a_2 b_1) \mathbf{e}_{12} + (a_2 b_3 - a_3 b_2) \mathbf{e}_{23} + (a_3 b_1 - a_1 b_3) \mathbf{e}_{31} . \tag{1.24}$$

The four-dimensional Minkowskian space Mi4 can be understood as Eu3 with an added dimension corresponding to the index 4. In this case, the ordering is usually taken cyclic in the indices 1,2,3 and the index 4 is written last as

$$\begin{aligned} \mathbf{a} \wedge \mathbf{b} &= (a_1 b_2 - a_2 b_1) \mathbf{e}_{12} + (a_2 b_3 - a_3 b_2) \mathbf{e}_{23} + (a_3 b_1 - a_1 b_3) \mathbf{e}_{31} \\ &\quad + (a_1 b_4 - a_4 b_1) \mathbf{e}_{14} + (a_2 b_4 - a_4 b_2) \mathbf{e}_{24} + (a_3 b_4 - a_4 b_3) \mathbf{e}_{34} . \end{aligned} \tag{1.25}$$

More generally, expressing Minkowskian vectors $\mathbf{a}_{\mathrm{M}}$ and dual vectors $\boldsymbol{\alpha}_{\mathrm{M}}$ as

$$\mathbf{a}_{\mathrm{M}} = \mathbf{a} + \mathbf{e}_4 a_4, \qquad \boldsymbol{\alpha}_{\mathrm{M}} = \boldsymbol{\alpha} + \varepsilon_4 \alpha_4, \tag{1.26}$$

where $\mathbf{a}$ and $\boldsymbol{\alpha}$ are vector and dual vector components in the Euclidean Eu3 space, the wedge product of two Minkowskian vectors can be expanded as

$$\mathbf{a}_{\mathrm{M}} \wedge \mathbf{b}_{\mathrm{M}} = (\mathbf{a} + \mathbf{e}_4 a_4) \wedge (\mathbf{b} + \mathbf{e}_4 b_4) = \mathbf{a} \wedge \mathbf{b} + (\mathbf{a} b_4 - \mathbf{b} a_4) \wedge \mathbf{e}_4 . \tag{1.27}$$

Thus, any bivector or dual bivector in the Mi4 space can be naturally expanded in the form

$$\mathbf{A}_{\mathrm{M}} = \mathbf{A} + \mathbf{a} \wedge \mathbf{e}_4, \qquad \boldsymbol{\Phi}_{\mathrm{M}} = \boldsymbol{\Phi} + \boldsymbol{\alpha} \wedge \varepsilon_4, \tag{1.28}$$

where $\mathbf{A}$, $\mathbf{a}$, $\boldsymbol{\Phi}$, and $\boldsymbol{\alpha}$ denote the respective Euclidean bivector, vector, dual bivector, and dual vector components.

For two-dimensional vectors the dimension of the bivectors is 1 and all bivectors can be expressed as multiples of a single basis element $\mathbf{e}_{12}$. Because for the three-dimensional vector space the bivector space has the dimension 3, bivectors have a close relation to vectors. In the Gibbsian vector algebra, where the wedge product is replaced by the cross product, bivectors are identified with vectors. In the four-dimensional vector space, bivectors form a six-dimensional space, and they can be represented in terms of a combination of a three-dimensional vector and bivector, each of dimension 3.

In terms of basis bivectors, respective expansions for the general bivector $\mathbf{A} = \sum_{i,j} A_{ij}\mathbf{e}_{ij}$ in spaces of dimension $n = 2, 3$, and 4 can be, respectively, written as

$$\mathbf{A} = A_{12}\mathbf{e}_{12}, \tag{1.29}$$

$$\mathbf{A} = A_{12}\mathbf{e}_{12} + A_{23}\mathbf{e}_{23} + A_{31}\mathbf{e}_{31}, \tag{1.30}$$

$$\mathbf{A} = A_{12}\mathbf{e}_{12} + A_{23}\mathbf{e}_{23} + A_{31}\mathbf{e}_{31} + A_{14}\mathbf{e}_{14} + A_{24}\mathbf{e}_{24} + A_{34}\mathbf{e}_{34}. \tag{1.31}$$

Similar expansions apply for the dual bivectors $\mathbf{\Phi} = \sum \Phi_{ij}\varepsilon_{ij}$. It can be shown that any bivector $\mathbf{A}$ in the case $n = 3$ can be expressed in the form of a simple bivector $\mathbf{A} = \mathbf{a} \wedge \mathbf{b}$ in terms of two vectors $\mathbf{a}, \mathbf{b}$. The proof is left as an exercise. This decomposition is not unique since, for example, we can write $(\mathbf{a} + \lambda\mathbf{b}) \wedge \mathbf{b}$ instead of $\mathbf{a} \wedge \mathbf{b}$ with any scalar λ without changing the bivector. On the other hand, for $n = 4$, any bivector can be expressed as a sum of two simple bivectors, in the form $\mathbf{A} = \mathbf{a} \wedge \mathbf{b} + \mathbf{c} \wedge \mathbf{d}$ in terms of four vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$. Again, this representation is not unique. The proof can be based on separating the fourth dimension as was done in (1.28).

1.3.3 Duality product

The duality product of a vector and a dual vector is straightforwardly generalized to that of a bivector and a dual bivector by defining the product for the reciprocal basis bivectors and dual bivectors as

$$\varepsilon_{12}|\mathbf{e}_{12} = 1, \quad \varepsilon_{12}|\mathbf{e}_{13} = 0, \quad \varepsilon_{13}|\mathbf{e}_{13} = 1, ... \tag{1.32}$$

and more generally

$$\varepsilon_J|\mathbf{e}_K = \delta_{JK}, \quad J = \{ij\}, \; K = \{k\ell\}. \tag{1.33}$$

Here, J and K are ordered bi-indices $(i < j, \; k < \ell)$ and the symbol δ_{JK} has the value 1 only when both $ij = k\ell$, otherwise it is zero. Thus, we can write

$$\delta_{JK} = \delta_{\{ij\}\{k\ell\}} = \delta_{ik}\delta_{j\ell}. \tag{1.34}$$

The corresponding definition for nonordered indices J, K has to take also into account that $\delta_{JK} = -1$ when $ij = \ell k$, in which case (1.34) is generalized to

$$\delta_{JK} = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}. \tag{1.35}$$