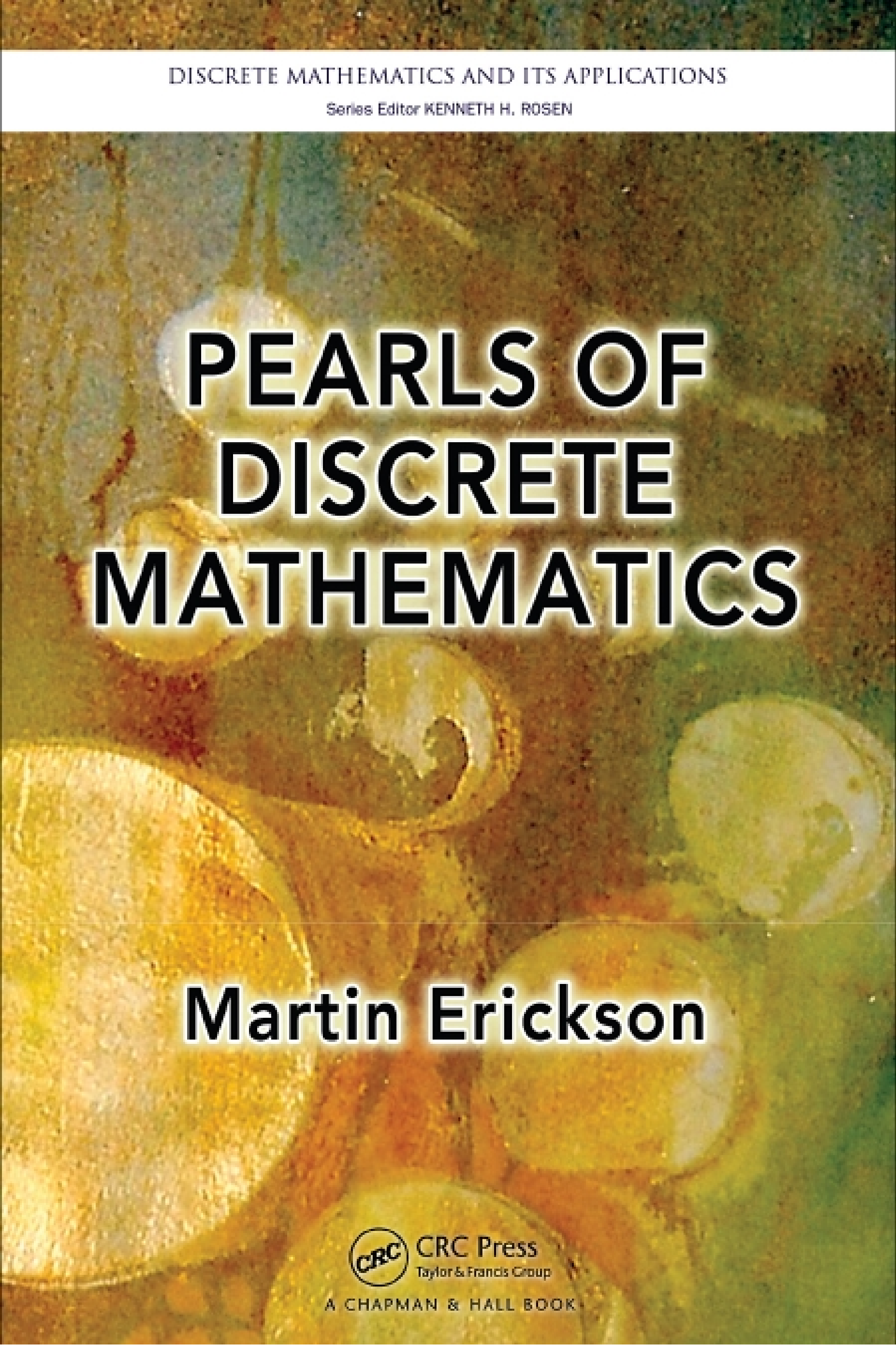
DISCRETE MATHEMATICS AND ITS APPLICATIONS
Series Editor KENNETH H. ROSEN
PEARLS OF DISCRETE MATHEMATICS
Martin Erickson
CRC Press
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A CHAPMAN & HALL BOOK

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Martin Erickson
Truman State University
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To my mother, Lorene Erickson

Contents

Preface

This book presents subjects suitable for a topics course or self-study in discrete mathematics. The focus is on representative, intriguing, and beautiful examples, problems, theorems, and proofs. Of course, the choice of coverage is personal and subjective, but I hope that the concepts I treat will be of interest to you and that you will find some as fascinating as I do.

Discrete structures (which are based on finite sets) comprise an area of mathematics that most people relate to naturally. How many ways can you choose an appetizer, a main course, and a dessert at dinner? How many walking routes can you take through town? How many lottery number combinations can you buy? These are typical counting problems in discrete mathematics. In our day-to-day mathematical lives, we often encounter counting problems or other types of problems involving discrete structures. We should learn the methods used to solve these problems.

Discrete structures play a central role in mathematics. They are intimately related to algebra, geometry, number theory, and combinatorics, and these relationships are illustrated with several of the pearls in this book. One needs only to look at the many journal titles in discrete mathematics (at least thirty in number) to see that this area is huge. The journal titles indicate connections between discrete mathematics and computing, information theory and codes, and probability. I think it's safe to say that all mathematicians and computer scientists would benefit from investigating the basic principles of discrete mathematics.

The world of discrete mathematics is like a mosaic or tapestry, with one pattern fitting into another and theories gradually emerging. I have taken an organic approach in this book, exploring concrete problems, introducing theory, and adding generalizations as we go.

Having taught mathematics for twenty-five years, I understand the importance of examples and exercises. Accordingly, I have kept examples and exercises at the forefront of the discussion. I've tried to arrange it so that each chapter features a particularly surprising, stunning, elegant, or unusual result. Included are mathematical items that don't appear in many books, such as the upward extension of Pascal's triangle, a recurrence relation for powers of Fibonacci numbers, the number of ways to make change for a million dollars, integer triangles, the period of Alcuin's sequence, Rook and Queen paths and the equivalent Nim and Wythoff's Nim games, the probability of a perfect bridge hand, random tournaments, a Fibonacci-like sequence of composite numbers, Shannon's theorems of information theory, higher-dimensional tic-tac-toe, animal achievement and avoidance games, and an algorithm for solving Sudoku puzzles and polycube packing problems. I introduce each chapter with a mathematical "teaser" or two to whet your appetite—mathematics can be engaging, inspiring, and even fun!

You will profit from doing the exercises, as a good deal of the mathematics is revealed there. The problems range in difficulty from easy to quite challenging. Exercises designated with a star ($\star$) are particularly difficult or require advanced mathematical background; exercises designated with a diamond ($\diamond$) require the use of a calculator or computer; exercises designated with a dagger ($\dagger$) are of theoretical importance. Hints or solutions to the exercises are provided in an appendix in the back of the book.

Thanks to the people who have kindly provided suggestions concerning this work: Robert Cacioppo, Robert Dobrow, Rodman Doll, Christine Erickson, Suren Fernando, David Garth, Joe Hemmeter, Daniel Jordan, Ken Price, Khang Tran, and Anthony Vazzana. Special thanks to Lorene Erickson for creating the cover artwork, *Spacescape V*. I would also like to thank the people at CRC Press, especially David Grubbs and Kenneth Rosen, for their help and encouragement in writing this book.

Part I

Counting: Basic

Chapter 1

Subsets of a Set

Abby has collected 100 *pennies. She offers Betty the choice of any or all of the pennies from her collection. The number of selections of pennies that Betty can make is* 1,267,650,600,228,229,401,496,703,205,376.

We begin with one of the simplest theorems of discrete mathematics. Denote by $\mathbf{N}$ the set of natural numbers, $\{1, 2, 3, \ldots\}$.

Theorem 1.1. Given $n \in \mathbf{N}$, a set with n elements has 2^n subsets.

Proof. Each element in the given n-element set can either be included or not included in a subset. Hence, there are

$$\underbrace{2 \times 2 \times 2 \times \cdots \times 2}_{n} = 2^n$$

choices in forming subsets. ∎

Notice that when choices are made independently, we *multiply* the numbers of choices. This principle is called the "product rule."

Example 1.2. How many subsets does the set $\{A, B, C\}$ have?

Solution: The set $\{A, B, C\}$ has $2^3 = 8$ subsets:

$$\emptyset,\ \{A\},\ \{B\},\ \{C\},\ \{A, B\},\ \{A, C\},\ \{B, C\},\ \{A, B, C\}.$$

∎

Example 1.3. How many subsets does a 100-element set have?

Solution: As stated in the teaser above, the number of subsets of a 100-element set is the colossal number

$$2^{100} = 1267650600228229401496703205376 \doteq 1.3 \times 10^{30}.$$

∎

Exercises

1. How many subsets does the set $\{A, B, C, D\}$ have? List the subsets.

2. At a certain dinner, there are four choices for the appetizer, two choices for the main course, and three choices for the dessert. How many different meals are possible?

◇3. Find the smallest integer n such that the number of subsets of an n-element set is greater than 10^{100} (a googol).

4. A teacher making a book display wants to showcase a novel, a history book, and a science book. There are four choices for the novel, two choices for the history book, and 10 choices for the science book. How many choices are possible for the three books?

5. A license consists of three digits (0 through 9), followed by a letter (A through Z), followed by another digit. How many different licenses are possible?

6. How many strings of 10 symbols are there in which the symbols may be 0, 1, or 2?

7. How many subsets of the set $\{a, b, c, d, e, f, g, h, i, j\}$ do not contain both a and b?

8. How many binary strings of length 99 are there such that the sum of the elements in the string is an odd number?

9. How many functions map the set $\{a, b, c\}$ to the set $\{w, x, y, z\}$?

10. How many functions map an n-element set to itself?

11. Let $X = \{1, 2, 3, \ldots, 2n\}$. How many functions map X to X such that each even number is mapped to an even number and each odd number is mapped to an odd number?

12. Is the result of Theorem 1.1 true for $n = 0$?

13. How many ways can you place a White King and a Black King on an 8×8 chessboard so that they don't attack each other? (A King attacks the squares horizontally, vertically, and diagonally adjacent to its own square.)

Chapter 2

Pascal's Triangle

Using Pascal's triangle, it is easy to calculate the number of subsets of a given set that have a certain size.

A *permutation* of a set is a selection of the elements of the set in some order. The number of permutations of n objects is $n!$, that is, the *factorial function* defined by

$$n! = n(n-1)(n-2)\dots 3\cdot 2\cdot 1, \text{ for } n \geq 1, \text{ and } 0! = 1.$$

Example 2.1. How many ways may 10 books be arranged on a shelf?

Solution: The number of arrangements is the number of permutations of 10 elements:

$$10! = 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 3628800.$$

■

More generally, the number of permutations of k objects from a set of n objects is

$$P(n,k) = n(n-1)(n-2)\dots(n-k+1) = \frac{n!}{(n-k)!}, \quad 0 \leq k \leq n.$$

Example 2.2. How many ways can four books from a set of 10 books be arranged on a shelf?

Solution: The number of arrangements is $P(10,4) = 10\cdot 9\cdot 8\cdot 7 = 5040$. ■

A *combination* from a set is a subset of the set, i.e., a selection of elements of the set in which the order of the selected elements doesn't matter. The number of k-element combinations from an n-element set is

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}, \quad 0 \leq k \leq n.$$

We set $C(n,k) = 0$ if $k < 0$ or $k > n$.

The numbers $C(n,k)$ are given in Pascal's triangle, named after Blaise

$$\begin{array}{c}
1\\
1 \quad 1\\
1 \quad 2 \quad 1\\
1 \quad 3 \quad 3 \quad 1\\
1 \quad 4 \quad 6 \quad 4 \quad 1\\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1\\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1\\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1\\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1\\
1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1\\
1 \quad 10 \quad 45 \quad 120 \quad 210 \quad 252 \quad 210 \quad 120 \quad 45 \quad 10 \quad 1\\
\vdots
\end{array}$$

FIGURE 2.1: Pascal's triangle.

Pascal (1623–1662). See Figure 2.1. The rows of Pascal's triangle are numbered 0, 1, 2, etc. from top to bottom, and the columns are numbered 0, 1, 2, etc. from left to right. Each non-1 entry of Pascal's triangle is equal to the sum of the two entries directly above it.

The entry in row n, column k of Pascal's triangle is $C(n, k)$. This number is the same as the *binomial coefficient*

$$\binom{n}{k}.$$

For example, entry 2 of row 6 is $C(6, 2) = \binom{6}{2} = 6!/(2!4!) = 15$.

Theorem 1.1 tells us that the sum of the numbers in row n of Pascal's triangle is 2^n. For example, the sum of the entries in the sixth row is $1 + 6 + 15 + 20 + 15 + 6 + 1 = 64 = 2^6$.

The simple rule that generates Pascal's triangle is a recurrence relation known as *Pascal's identity*:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \quad 1 \le k \le n-1, \qquad \binom{n}{0} = 1, \ \binom{n}{n} = 1.$$

Pascal's identity has a counting proof. The binomial coefficient $\binom{n}{k}$ is the number of k-element subsets of the set $\{1, \ldots, n\}$. Each such subset either contains the element n or does not contain n. The number of k-element subsets that contain n is $\binom{n-1}{k-1}$. The number of k-element subsets that don't contain n is $\binom{n-1}{k}$.

Binomial coefficients get their name because they are the coefficients of a binomial expansion. That is, $\binom{n}{k}$ is the coefficient of $a^{n-k}b^k$ in $(a+b)^n$. The binomial theorem says just that.

Theorem 2.3 (Binomial Theorem)**.**

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k, \quad n \geq 0.$$

Example 2.4. Give the expansion of $(a+b)^6$.

Solution: By the binomial theorem, the expansion of $(a+b)^6$ is

$$\binom{6}{0}a^6b^0 + \binom{6}{1}a^5b^1 + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \binom{6}{5}a^1b^5 + \binom{6}{6}a^0b^6$$
$$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$$

■

Armed with the binomial theorem, we can give a second proof of Theorem 1.1. Set $a = 1$ and $b = 1$, and we get

$$2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

Thus, there are 2^n subsets of an n-element set.

Exercises

1. A teacher has eight books to put on a shelf. How many different orderings of the books are possible?

2. A student has 10 books but only room for six of them on a shelf. How many permutations of the books are possible on the shelf?

◇3. Find the smallest integer n such that $n! > 10^{100}$ (a googol).

4. A couple plans to visit three selected cities in Germany, followed by four selected cities in France, followed by five selected cities in Spain. In how many ways can the couple order their itinerary?

5. You have three small glasses, four medium-size glasses, and five large glasses. If glasses of the same size are indistinguishable, how many ways can you arrange the glasses in a row?

6. A librarian wants to arrange four astronomy books, five medical books, and six religious books on a shelf. Books of the same category should be grouped together, but otherwise the books may be put in any order. How many orderings are possible?

7. In how many ways can you arrange the letters of the word RHODODENDRUM?

8. How many one-to-one functions are there from the set $\{a, b, c\}$ to the set $\{t, u, v, w, x, y, z\}$?

9. Let X be an n-element set, where $n \in \mathbf{N}$. How many functions from X to X are not one-to-one?

10. Find a formula for the number of different binary relations possible on a set of n elements, where $n \in \mathbf{N}$.

11. Professor Bumble teaches five different classes, A, B, C, D, and E. He prepared a different lecture for each class today, but he gave some or all of the lectures to the wrong classes. He knows that class A received the wrong lecture. In how many different orders can Professor Bumble have given his lectures today?

12. A singer plans to perform three songs from a repertoire of 12 songs. How many different programs are possible if there are two songs, A and B, that cannot both be performed?

13. A student decides to take three classes from a set of 10. In how many ways may she do this?

14. In a certain lottery, a contestant must choose six numbers from the set 1, 2, ..., 44. How many combinations are possible?

15. Evaluate $\binom{20}{10}$.

16. What is the coefficient of $a^{10}b^{10}$ in the expansion of $(a + b)^{20}$?

17. Give the expansion of $(a + b)^{10}$.

18. Give simple formulas for $\binom{n}{2}$ and $\binom{n}{3}$.

19. What is the coefficient of x^{10} in $(1 + x^2)^{20}$?

20. What is the coefficient of x^7 in $(1 - 2x)^{10}$?

21. What is the constant in the expansion of $(x + 1/x)^{20}$?

22. What is the constant in the expansion of $(x^4 + 1/x)^{20}$?

23. Explain the formula $C(n, k) = P(n, k)/k!$.

24. Give an algebraic proof of Pascal's identity.

†25. Prove the binomial theorem.

26. Professor Bumble doesn't remember how many students are in his honors mathematics class. But he does remember that there are 924 ways to divide the class into two equal-size groups of students. Help Professor Bumble determine how many students are in his class.

27. A pointer starts at 0 on the real number line and moves right or left one unit at each step. Let n and k be positive integers. How many different paths of k steps terminate at the integer n?

◇28. Use the recurrence relation for Pascal's triangle to compute the value of the binomial coefficient $\binom{100}{50}$.

29. Suppose that n lines are given in the plane in "general position" (no two parallel and no three concurrent at a point). Into how many regions is the plane partitioned?

30. Suppose that n planes are given in three-dimensional space in "general position" (no two parallel, no three concurrent in a line, and no four concurrent in a point). Into how many regions is space partitioned?

Chapter 3

Binomial Coefficient Identities

Every finite nonempty set has as many subsets with an even number of elements as subsets with an odd number of elements.

Consider again the subsets of the three-element set $\{A, B, C\}$. Half of these subsets have an even number of elements and half an odd number of elements:

even number of elements	odd number of elements
$\emptyset$	$\{A\}$
$\{A, B\}$	$\{B\}$
$\{A, C\}$	$\{C\}$
$\{B, C\}$	$\{A, B, C\}$.

We claim that this property holds for every finite nonempty set.

Proposition 3.1. Given $n \geq 1$, half of the subsets of an n-element set have an even number of elements and half have an odd number of elements.

Proof. Let $a = 1$ and $b = -1$ in the binomial theorem. Then

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = (-1 + 1)^n = 0^n = 0, \quad n \geq 1.$$

Putting the summands associated with even k on one side of the relation and those associated with odd k on the other side, we obtain

$$\sum_{k \text{ odd}} \binom{n}{k} = \sum_{k \text{ even}} \binom{n}{k}.$$

■

For n odd, this assertion follows trivially from the symmetry of the binomial coefficients, that is, $\binom{n}{k} = \binom{n}{n-k}$. We give a counting argument valid for all $n \geq 1$. Let $X = \{1, 2, 3, \ldots, n\}$ and

$$\begin{aligned}
\mathcal{A} &= \{S \subseteq X : |S| \text{ is even and } 1 \in S\} \\
\mathcal{B} &= \{S \subseteq X : |S| \text{ is odd and } 1 \in S\} \\
\mathcal{C} &= \{S \subseteq X : |S| \text{ is even and } 1 \notin S\} \\
\mathcal{D} &= \{S \subseteq X : |S| \text{ is odd and } 1 \notin S\}.
\end{aligned}$$

There is a one-to-one correspondence between $\mathcal{A}$ and $\mathcal{D}$: simply remove 1 from every element of $\mathcal{A}$ to form an element of $\mathcal{D}$. (What is the inverse?) Similarly, there is a one-to-one correspondence between $\mathcal{B}$ and $\mathcal{C}$. Hence $|\mathcal{A}| = |\mathcal{D}|$ and $|\mathcal{B}| = |\mathcal{C}|$, and it follows that

$$|\mathcal{A}| + |\mathcal{C}| = |\mathcal{B}| + |\mathcal{D}|.$$

Let's explore a few more binomial coefficient identities.

Example 3.2. What is the sum $\sum_{k=0}^{n} \binom{n}{k}^2$?

Solution: We work out some examples using Pascal's triangle:

$$\begin{aligned}
&n = 1: \; 1^2 = 1 \\
&n = 2: \; 1^2 + 1^2 = 2 \\
&n = 3: \; 1^2 + 2^2 + 1^2 = 6 \\
&n = 4: \; 1^2 + 3^2 + 3^2 + 1^2 = 20 \\
&n = 5: \; 1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 70.
\end{aligned}$$

We recognize these sums as central binomial coefficients, and we make the conjecture that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Typically, the mathematical process consists of working examples, looking for patterns, making conjectures, and proving the conjectures. Let's try to prove our conjecture.

We rewrite our conjecture as follows:

$$\binom{n}{0}\binom{n}{0} + \binom{n}{1}\binom{n}{1} + \binom{n}{2}\binom{n}{2} + \cdots + \binom{n}{n}\binom{n}{n} = \binom{2n}{n}.$$

We know that the right side counts the ways of selecting n numbers from the set $\{1, 2, 3, \ldots, 2n\}$. Why is this counted by the left side? Rewrite a little, using symmetry:

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0} = \binom{2n}{n}.$$

Now the truth of the identity is clear. The right side counts the number of n-element subsets of $\{1, 2, 3, \ldots, 2n\}$. The left side counts the same thing, where, for $0 \leq k \leq n$, the term $\binom{n}{k}\binom{n}{n-k}$ counts the number of subsets in which k elements are chosen from the set $\{1, \ldots, n\}$ and $n-k$ elements are chosen from the set $\{n+1, \ldots, 2n\}$. ■

The identity of Example 3.2 has a counting interpretation in terms of *lattice paths*, i.e., paths along the lines of an $n \times n$ grid. The binomial coefficient $\binom{2n}{n}$ is the number of northeast paths that start at the southwest corner of the grid and stop at the northeast corner. Each path has length $2n$ and is determined by a sequence of n "easts" and n "norths" in some order. The summation $\sum_{k=0}^{n} \binom{n}{k}^2$ counts the paths according to their intersection with the main southeast diagonal of the grid. The number of paths that cross the diagonal at the point k units east of the starting point is $\binom{n}{k}^2$, where $0 \leq k \leq n$.

Both Pascal's identity and the identity of Example 3.2 are generalized by *Vandermonde's identity*, credited to Alexandre-Théophile Vandermonde (1735–1796):

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}, \quad m, n, k \geq 0.$$

The binomial coefficient $\binom{m+n}{k}$ is the number of k-element subsets of the set $\{1, \ldots, m+n\}$. The number of such subsets that contain i elements from the set $\{1, \ldots, m\}$ and $k-i$ elements from the set $\{m+1, \ldots, m+n\}$ is $\binom{m}{i}\binom{n}{k-i}$. The summation $\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}$ counts these subsets for $0 \leq i \leq k$. Letting $m = 1$, and changing n to $n-1$, the relation becomes Pascal's identity. Letting $k = m = n$, we obtain the identity from Example 3.2.

The next identity is often useful (e.g., see Chapter 13).

Proposition 3.3 ("Subcommittee Identity")**.** For $0 \leq j \leq k \leq n$, we have

$$\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}.$$

Proof. Both expressions count the number of ways to choose, from n people, a committee of size k and a subcommittee of size j. ■