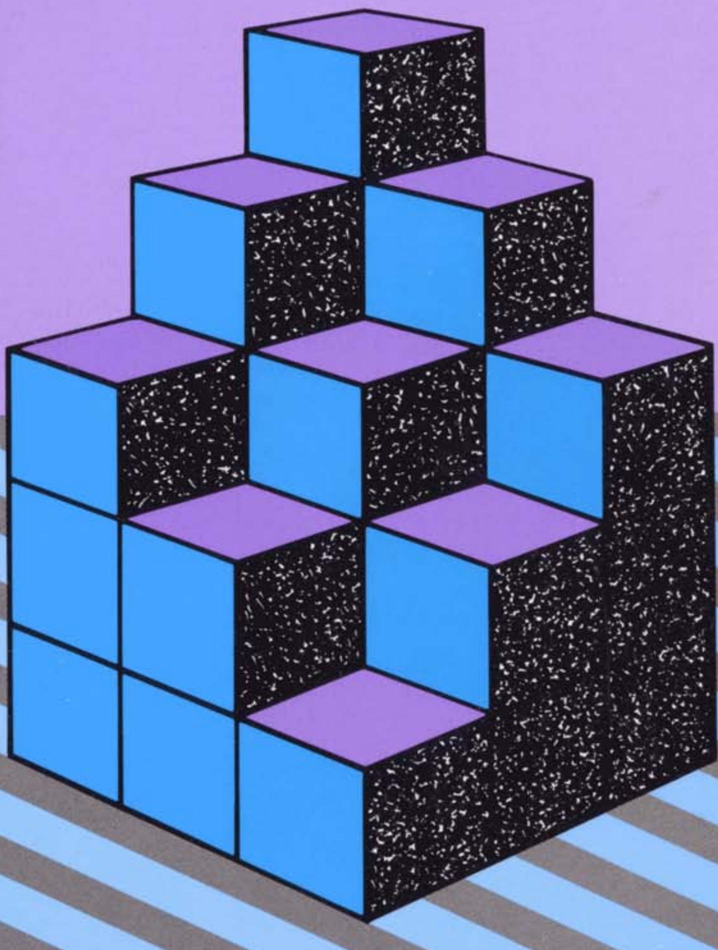
THE SECOND SCIENTIFIC AMERICAN BOOK OF MATHEMATICAL PUZZLES AND DIVERSIONS

MARTIN GARDNER



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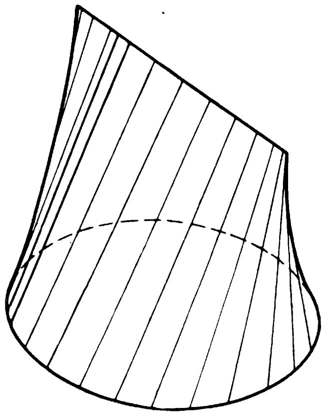

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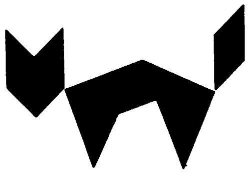

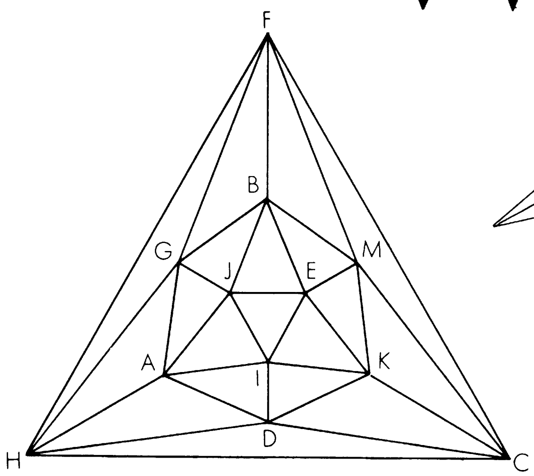

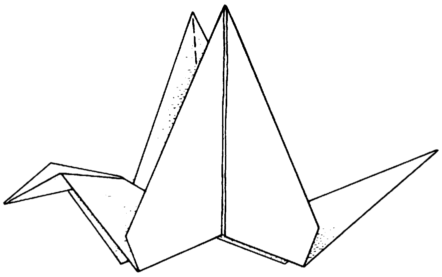

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MARTIN GARDNER

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A NEW SELECTION:
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from Digital Roots and Dudeney Puzzles to
the Diabolic Square, from the Golden Ratio
to the Generalized Ham Sandwich Theorem.
With mathematical commentaries by Mr. Gardner,
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references for further reading
and, of course, solutions.

With a new Postscript by the author

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For J. H. G.
who likes to tackle puzzles
big enough to walk upon

CONTENTS

INTRODUCTION

SINCE THE APPEARANCE *of the first* Scientific American Book of Mathematical Puzzles & Diversions, *in 1959, popular interest in recreational mathematics has continued to increase. Many new puzzle books have been printed, old puzzle books reprinted, kits of recreational math materials are on the market, a new topological game (see Chapter 7) has caught the fancy of the country's youngsters, and an excellent little magazine called* Recreational Mathematics *has been started by Joseph Madachy, a research chemist in Idaho Falls. Chessmen — those intellectual status symbols — are jumping all over the place, from TV commercials and magazine advertisements to Al Horowitz's lively chess corner in* The Saturday Review *and the knight on Paladin's holster and have-gun-will-travel card.*

This pleasant trend is not confined to the U.S. A classic four-volume French work, Récréations Mathématiques, *by Edouard Lucas, has been reissued in France in paperbacks. Thomas H. O'Beirne, a Glasgow mathematician, is writing a splendid puzzle column in a British science journal. In the U.S.S.R. a handsome 575-page collection of puzzles, assembled by mathematics teacher Boris Kordemski, is selling in Russian and Ukrainian editions. It is all, of course, part of a world-wide boom in math — in turn a reflection of the increasing demand for skilled mathematicians to meet the incredible needs of the new triple age of the atom, spaceship and computer.*

The computers are not replacing mathematicians; they *are breeding them. It may take a computer less than twenty seconds to solve a thorny problem, but it may have taken a group of mathematicians many months to program the problem. In addition, scientific research is becoming more and more dependent on the mathematician for important breakthroughs in theory. The relativity revolution, remember, was the work of a man who had no experience in the laboratory. At the moment, atomic scientists are thoroughly befuddled by the preposterous properties of some thirty different fundamental particles; "a vast jumble of odd dimensionless numbers," as J. Robert Oppenheimer has described them, "none of them understandable or derivable, all with an insulting lack of obvious meaning." One of these days a great creative mathematician, sitting alone and scribbling on a piece of paper, or shaving, or taking his family on a picnic, will experience a flash of insight. The particles will spin into their appointed places, rank on rank, in a beautiful pattern of unalterable law. At least, that is what the particle physicists* hope *will happen. Of course the great puzzle solver will draw on laboratory data, but the chances are that he will be, like Einstein, primarily a mathematician.*

Not only in the physical sciences is mathematics battering down locked doors. The biological sciences, psychology and the social sciences are beginning to reel under the invasion of mathematicians armed with strange new statistical techniques for designing experiments, analyzing data, predicting probable results. It may still be true that if the President of the United States asks three economic advisers to study an important question, they will report back with four different opinions; but it is no longer absurd to imagine a distant day when economic disagreements can be settled by mathematics in a way that is not subject to the usual dismal disputes. In the cold light of modern economic theory the conflict between *socialism and capitalism is rapidly becoming, as Arthur Koestler has put it, as naïve and sterile as the wars in Lilliput over the two ways to break an egg. (I speak only of the economic debate; the conflict between democracy and totalitarianism has nothing to do with mathematics.)*

But those are weighty matters and this is only a book of amusements. If it has any serious purpose at all it is to stimulate popular interest in mathematics. Such stimulation is surely desirable, if for no other reason than to help the layman understand what the scientists are up to. And they are up to plenty.

I would like to express again my gratitude to the publisher, editors and staff of Scientific American, *the magazine in which these chapters first appeared; to my wife for assistance in many ways; and to the hundreds of friendly readers who continue to correct my errors and suggest new material. I would like also to thank, for her expert help in preparing the manuscript, Nina Bourne of Simon and Schuster.*

MARTIN GARDNER

CHAPTER ONE

The Five Platonic Solids

A REGULAR POLYGON is a plane figure bounded by straight lines, with equal sides and equal interior angles. There is of course an infinite number of such figures. In three dimensions the analog of the regular polygon is the regular polyhedron: a solid bounded by regular polygons, with congruent faces and congruent interior angles at its corners. One might suppose that these forms are also infinite, but in fact they are, as Lewis Carroll once expressed it, "provokingly few in number." There are only five regular convex solids: the regular tetrahedron, hexahedron (cube), octahedron, dodecahedron and icosahedron [*see Fig. 1*].

The first systematic study of the five regular solids appears to have been made by the ancient Pythagoreans. They believed that the tetrahedron, cube, octahedron and icosahedron respectively underlay the structure of the traditional

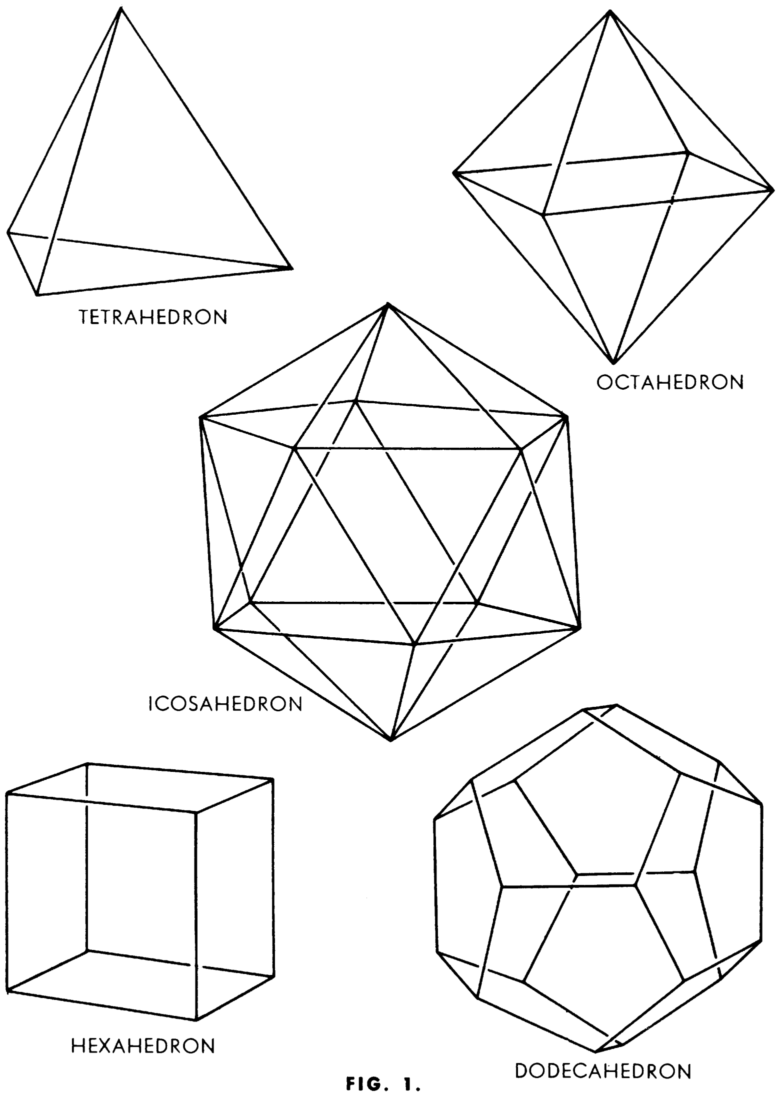

FIG. 1.

The five Platonic solids. The cube and octahedron are "duals" in the sense that if the centers of all pairs of adjacent faces on one are connected by straight lines, the lines form the edges of the other. The dodecahedron and icosahedron are dually related in the same way. The tetrahedron is its own dual.

four elements: fire, earth, air and water. The dodecahedron was obscurely identified with the entire universe. Because these notions were elaborated in Plato's *Timaeus,* the regular polyhedrons came to be known as the Platonic solids. The beauty and fascinating mathematical properties of these five forms haunted scholars from the time of Plato through the Renaissance. The analysis of the Platonic solids provides the climactic final book of Euclid's *Elements.* Johannes Kepler believed throughout his life that the orbits of the six planets known in his day could be obtained by nesting the five solids in a certain order within the orbit of Saturn. Today the mathematician no longer views the Platonic solids with mystical reverence, but their rotations are studied in connection with group theory and they continue to play a colorful role in recreational mathematics. Here we shall quickly examine a few diversions in which they are involved.

There are four different ways in which a sealed envelope can be cut and folded into a tetrahedron. The following is perhaps the simplest. Draw an equilateral triangle on both sides of one end of an envelope [*see Fig. 2*]. Then cut through

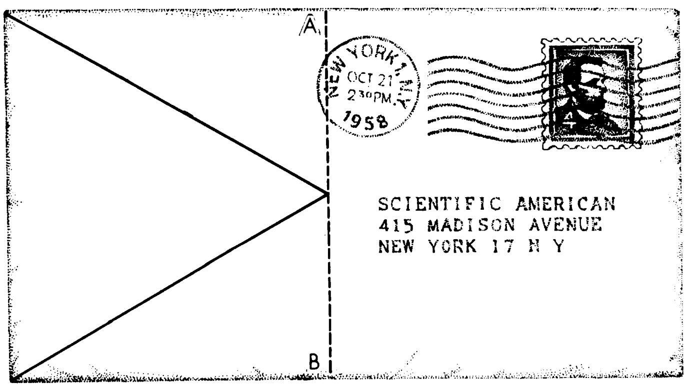

FIG. 2.
How a sealed envelope can be cut for folding into a tetrahedron.

both layers of the envelope as indicated by the broken line and discard the right-hand piece. By creasing the paper along the sides of the front and back triangles, points A and B are brought together to form the tetrahedron.

Figure 3 shows the pattern for a tantalizing little puzzle currently marketed in plastic. You can make the puzzle yourself by cutting two such patterns out of heavy paper. (All the line segments except the longer one have the same length.) Fold each pattern along the lines and tape the edges to make the solid shown. Now try to fit the two solids together to make a tetrahedron. A mathematician I know likes to annoy his friends with a practical joke based on this puzzle. He bought two sets of the plastic pieces so that he

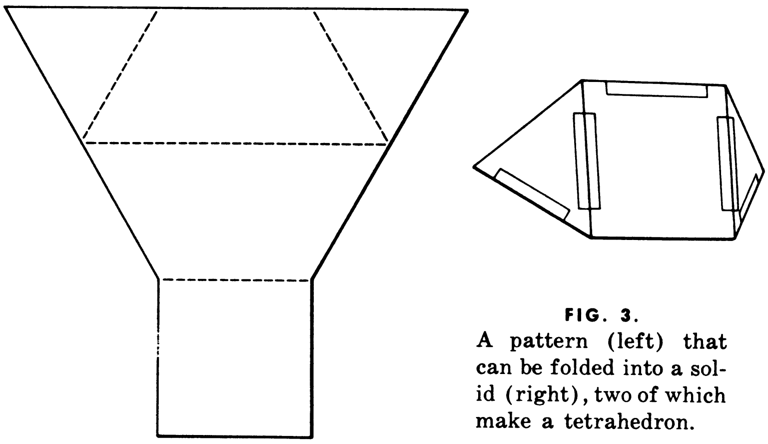

FIG. 3.
A pattern (left) that can be folded into a solid (right), two of which make a tetrahedron.

could keep a third piece concealed in his hand. He displays a tetrahedron on the table, then knocks it over with his hand and at the same time releases the concealed piece. Naturally his friends do not succeed in forming the tetrahedron out of the *three* pieces.

Concerning the cube I shall mention only an electrical puzzle and the surprising fact that a cube can be passed through a hole in a smaller cube. If you will hold a cube so that one corner points directly toward you, the edges outlining a hexagon, you will see at once that there is ample space for a square hole that can be slightly larger than the face of the cube itself. The electrical puzzle involves the network depicted in Figure 4. If each edge of the cube has a

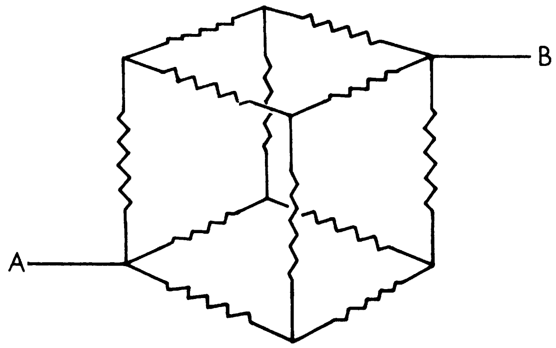

FIG. 4.
An electrical-network puzzle.

resistance of one ohm, what is the resistance of the entire structure when current flows from A to B? Electrical engineers have been known to produce pages of computations on this problem, though it yields easily to the proper insight.

All five Platonic solids have been used as dice. Next to the cube the octahedron seems to have been the most popular. The pattern shown in Figure 5, its faces numbered as indicated, will fold into a neat octahedron whose open edges can be closed with transparent tape. The opposite sides of this die, as in the familiar cubical dice, total seven. Moreover, a pleasant little mind-reading stunt is made possible by this arrangement of digits. Ask someone to think of a number from 0 to 7 inclusive. Hold up the octahedron so that he sees only the faces 1, 3, 5 and 7, and ask him if he sees his chosen number. If he says "Yes," this answer has a key value of 1. Turn the solid so that he sees faces 2, 3, 6 and 7, and ask the question again. This time "Yes" has the value of 2. The final question is asked with the solid turned so that

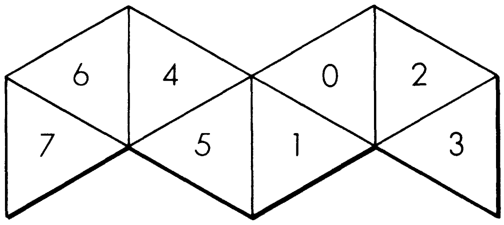

FIG. 5.
A strip to make an octahedral die.

he sees 4, 5, 6 and 7. Here a "Yes" answer has the value of 4. If you now total the values of his three answers you obtain the chosen number, a fact that should be easily explained by anyone familiar with the binary system. To facilitate finding the three positions in which you must hold the solid, simply mark in some way the three corners which must be pointed toward you as you face the spectator.

There are other interesting ways of numbering the faces of an octahedral die. It is possible, for example, to arrange the digits 1 through 8 in such a manner that the total of the four faces around each corner is a constant. The constant must be 18, but there are three distinct ways (not counting rotations and reflections) in which the faces can be numbered in this fashion.

An elegant way to construct a dodecahedron is explained in Hugo Steinhaus's book *Mathematical Snapshots.* Cut from heavy cardboard two patterns like the one pictured at left in Figure 6. The pentagons should be about an inch on a side. Score the outline of each center pentagon with the point of a knife so that the pentagon flaps fold easily in one direction. Place the patterns together as shown at right in the illustration so that the flaps of each pattern fold toward the others. Weave a rubber band alternately over and under the projecting ends, keeping the patterns pressed flat. When you release the pressure, the dodecahedron will spring magically into shape.

If the faces of this model are colored, a single color to each face, what is the minimum number of colors needed to make sure that no edge has the same color on both sides? The answer is four, and it is not difficult to discover the four different ways that the colors can be arranged (two are mirror images of the other two). The tetrahedron also requires four colors, there being two arrangements, one a reflection of the other. The cube needs three colors and the octahedron two, each having only one possible arrangement. The icosahedron calls for three colors; here there are no less than 144 different patterns, only six of which are identical with their mirror images.

If a fly were to walk along the 12 edges of an icosahedron, traversing each edge at least once, what is the shortest dis-

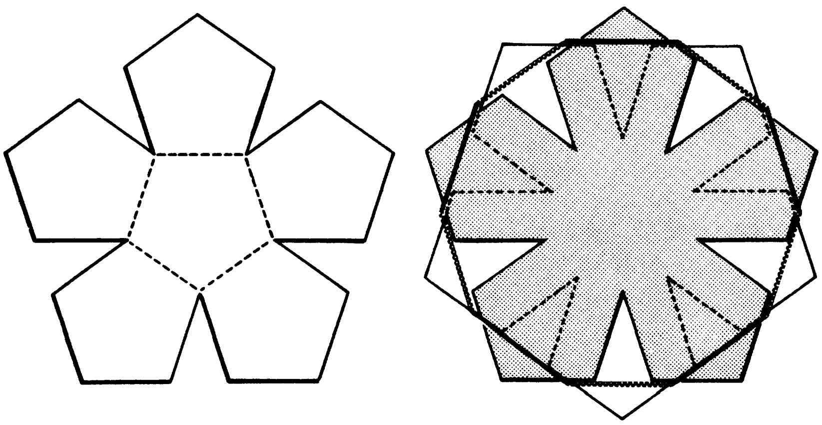

FIG. 6.
Two identical patterns are fastened together with a rubber band to make a pop-up dodecahedron.

tance it could travel? The fly need not return to its starting point, and it would be necessary for it to go over some edges twice. (Only the octahedron's edges can be traversed without retracing.) A plane projection of the icosahedron [*Fig. 7*] may be used in working on this problem, but one must remember that each edge is one unit in length. (I have been unable to resist concealing a laconic Christmas greeting in the way the corners of this diagram are labeled. It is not necessary to solve the problem in order to find it.)

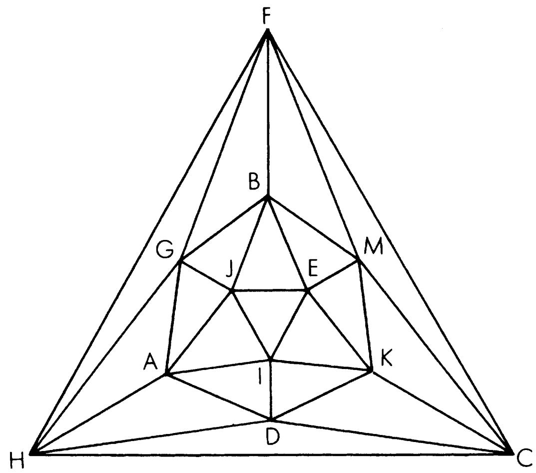

FIG. 7.
A plane projection of an icosahedron.

In view of the fact that cranks persist in trying to trisect the angle and square the circle long after these feats have been proved impossible, why has there been no comparable effort to find more than five regular polyhedrons? One reason is that it is quite easy to "see" that no more are possible. The following simple proof goes back to Euclid.

A corner of a polyhedron must have at least three faces. Consider the simplest face: an equilateral triangle. We can form a corner by putting together three, four or five such triangles. Beyond five, the angles total 360 degrees or more and therefore cannot form a corner. We thus have three possible ways to construct a regular convex solid with triangular faces. Three and only three squares will similarly form a corner, indicating the possibility of a regular solid with square faces. The same reasoning yields one possibility with three pentagons at each corner. We cannot go beyond the pentagon, because when we put three hexagons together at a corner, they equal 360 degrees.

This argument does not prove that five regular solids can be constructed, but it does show clearly that no more than five are possible. More sophisticated arguments establish that there are six regular polytopes, as they are called, in four-dimensional space. Curiously, in every space of more than four dimensions there are only three regular polytopes: analogs of the tetrahedron, cube and octahedron.

A moral may be lurking here. There is a very real sense in which mathematics limits the kinds of structures that can exist in nature. It is not possible, for example, that beings in another galaxy gamble with dice that are regular convex polyhedra of a shape unknown to us. Some theologians have been so bold as to contend that not even God himself could construct a sixth Platonic solid in three-dimensional space. In similar fashion, geometry imposes unbreakable limits on the varieties of crystal growth. Some day physicists may even discover mathematical limitations to the number of fundamental particles and basic laws. No one of course has any notion of how mathematics may, if indeed it does, restrict the nature of structures that can be called "alive." It is conceivable, for example, that the properties of carbon compounds are absolutely essential for life. In any case, as humanity braces itself for the shock of finding life on other planets, the Platonic solids serve as ancient reminders that there may be fewer things on Mars and Venus than are dreamt of in our philosophy.

ANSWERS

THE TOTAL resistance of the cubical network is 5/6 ohm. If the three corners closest to A are short-circuited together, and the same is done with the three corners closest to B, no current will flow in the two triangles of short circuits because each connects equipotential points. It is now easy to see that there are three one-ohm resistors in parallel between A and the nearest triangle (resistance 1/3 ohm), six in parallel between the triangles (1/6 ohm), and three in parallel between the second triangle and B (1/3 ohm), making a total resistance of 5/6 ohm.

C. W. Trigg, discussing the cubical-network problem in the November-December 1960 issue of *Mathematics Magazine,* points out that a solution for it may be found in *Magnetism and Electricity,* by E. E. Brooks and A. W. Poyser, 1920. The problem and the method of solving it can be easily extended to networks in the form of the other four Platonic solids.

The three ways to number the faces of an octahedron so that the total around each corner is 18 are: 6, 7, 2, 3 clockwise (or counterclockwise) around one corner, and 1, 4, 5, 8 around the opposite corner (6 adjacent to 1, 7 to 4 and so on); 1, 7, 2, 8 and 4, 6, 3, 5; and 4, 7, 2, 5 and 6, 1, 8, 3. See W. W. Rouse Ball's *Mathematical Recreations and Essays,* Chapter 7, for a simple proof that the octahedron is the only one of the five solids whose faces can be numbered so that there is a constant sum at each corner.

The shortest distance the fly can walk to cover all edges of an icosahedron is 35 units. By erasing five edges of the solid (for example, edges FM, BE, JA, ID and HC) we are left with a network that has only two points, G and K, where an odd number of edges come together. The fly can therefore traverse this network by starting at G and going to K without retracing an edge — a distance of 25 units. This is the longest distance it can go without retracing. Each erased edge can now be added to this path, whenever the fly reaches it, simply by traversing it back and forth. The five erased edges, each gone over twice, add 10 units to the path, making a total of 35.

The Christmas message conveyed by the letters is "Noel" (no "L").

CHAPTER TWO

Tetraflexagons

HEXAFLEXAGONS are diverting six-sided paper structures that can be "flexed" to bring different surfaces into view. They are constructed by folding a strip of paper as explained in the first *Scientific American Book of Mathematical Puzzles and Diversions*. Close cousins to the hexaflexagons are a wide variety of four-sided structures which may be grouped loosely under the term tetraflexagon.

Hexaflexagons were invented in 1939 by Arthur H. Stone, then a graduate student at Princeton University and now a lecturer in mathematics at the University of Manchester in England. Their properties have been thoroughly investigated; indeed, a complete mathematical theory of hexaflexigation has been developed. Much less is known about tetraflexagons. Stone and his friends (notably John W. Tukey, now a well-known topologist) spent considerable time folding and analyzing these four-sided forms, but did not succeed in developing a comprehensive theory that would cover all their discordant variations. Several species of tetraflexagon are nonetheless intensely interesting from the recreational standpoint.

Consider first the simplest tetraflexagon, a three-faced structure which can be called the tri-tetraflexagon. It is easily folded from the strip of paper shown in Figure 8 (8a is the front of the strip; 8b, the back). Number the small squares on each side of the strip as indicated, fold both ends inward (8c) and join two edges with a piece of transparent tape (8d). Face 2 is now in front; face 1 is in back. To flex the structure, fold it back along the vertical center line of face 2. Face 1 will fold into the flexagon's interior as face 3 flexes into view.

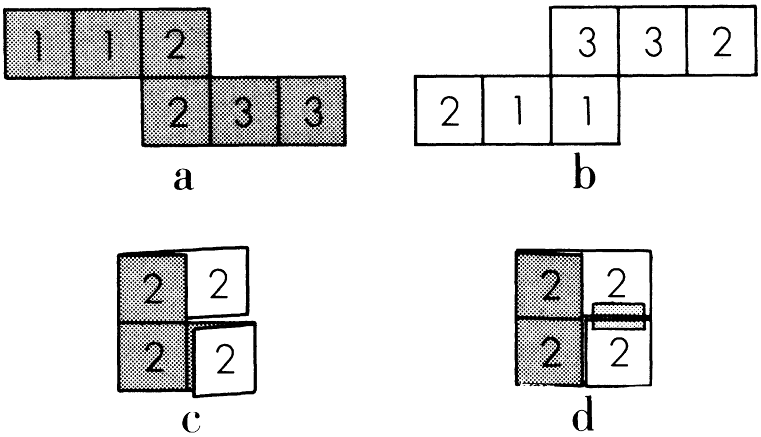

FIG. 8.
How to make a tri-tetraflexagon.

Stone and his friends were not the first to discover this interesting structure; it has been used for centuries as a double-action hinge. I have on my desk, for instance, two small picture frames containing photographs. The frames

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